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The GRW model and Bell-like inequalities*

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Abstract
The basics of quantum mechanics with spontaneous localization (GRW model) are rediscussed in the framework of a quantum stochastic process introduced by Ford and Lewis and originated by instantaneous fuzzy space-localization processes superimposed upon an otherwise reversible Schrödinger time evolution.

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1. Introduction

The GRW model consists essentially, in the introduction into the quantum formalism, of a mechanism of spontaneous space-decorrelation during the time evolution of any system. Decorrelation effects result practically absent for microscopic bodies and become instead effective for macroscopic particles, where the attribute *macroscopic* is understood in terms of the phenomenological parameters of the theory. As a consequence, quantum and classical evolutions turn out to be reconciled.

In its first version the GRW approach has been formulated as a reduction model in terms of an appropriate master equation for the time evolution of the quantum system statistical operator ρ [1]. Strictly speaking, the space-decorrelation effects were obtained by multiplication of the matrix elements of ρ in position representation by a Gaussian function, which, following John Bell [2], we will call *jump function*.[3] The late John Bell also pointed out that a version of the model in terms of a stochastic time evolution of wavefunctions instead of density matrices was more appealing from a physical point of view and, at the same time, expressed some doubts on the arbitrariness of the choice of a Gaussian function as the source of spontaneous localizations in position.

The first remark by John Bell was particularly important, because, as is well known, a same density matrix describes infinitely many different mixtures. Therefore, the vanishing of the off-diagonal matrix elements in position representation of a statistical operator is not sufficient

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[3] As a matter of fact, Bell called *jump function* the function multiplying the wavefunction. We extend here his terminology.

in itself to ensure the absence of space-correlations at the level of the pure components. It is thus necessary that the reduction processes affect every component of the ensemble represented by ρ.

As a matter of fact, the two remarks by Bell could be overcome [3] in such a way that the theory turned out to be based on a process which happens at the wavefunction level. Such a process obeys a set of general physical and mathematical assumptions whereby no particular choice of jump function is necessary[4].

In the same year when the GRW model appeared, Ford and Lewis [5], studying the evolution of a stochastic system in the framework of quantum mechanics, introduced a description of such a process in terms of fuzzy random-position measurements, ascribed to the coupling of the system with an external heat bath. In order to show that the effects of the repeated measurements always perturb the dynamics in the quantum case, the authors introduced inequalities[5] for certain joint probability distributions that are always fulfilled by classical time evolutions, being instead in general violated by quantum ones.

The analysis of the mechanism of violation [6] reveals its essential dependence on the system correlations and upon their behaviour during the Schrödinger time evolution. Thus, it is a natural guess that using the GRW evolution instead of the Schrödinger one should retain the violation in the case of microscopic systems, when the decorrelation is negligible, and eliminate it for macroscopic particles.

The aim of this paper is to review the results of [3, 6], revisiting the GRW model within the framework of the Ford–Lewis quantum stochastic process [5] in a way that clearly shows the spontaneous localizations in action.

2. Quantum mechanics with spontaneous localization revisited

The so-called quantum mechanics with spontaneous localization (GRW Model) is based on the assumption that any particle, whether isolated or constituent of any physical system, besides evolving through the standard Hamiltonian dynamics, is subjected at random times to spatial localization processes. The basic requisites for the theory to work and be physically consistent are:

(1) The localization effects on any normalized wavefunction $\psi(q)$ correspond to a collapse around the position $\bar{q}$ described as follows:

$$\psi(q) \to \psi_{\bar{q}}(q) = \phi_{\bar{q}}(q)/\|\phi_{\bar{q}}\|, \qquad \text{where} \quad \phi_{\bar{q}}(q) = \widetilde{f}(q-\bar{q})\psi(q). \tag{2.1}$$

(2) The function $\widetilde{f}(x)$ is asked to enjoy the following mathematical properties:
   - (a) it has to be real and positive, with continuous derivatives;
   - (b) it has a local maximum at $x = 0$ and goes to zero for $x \to \pm\infty$;
   - (c) since in the process only the difference $|q - \bar{q}|$ is important, it must be a symmetric function of $\bar{q}$, i.e., $\widetilde{f}(x)$ must be an even function of x.

(3) The process in (2.1) is assumed to occur with a constant time probability density λ, and with a spatial probability density of occurring at point $\bar{q}$ given by

$$\|\phi_{\bar{q}}\|^2 = \int_{-\infty}^{+\infty} \mathrm{d}q\, \widetilde{f}^2(q-\bar{q})|\psi(q)|^2, \qquad \text{with} \quad \int_{-\infty}^{+\infty} \mathrm{d}\bar{q}\, \|\phi_{\bar{q}}\|^2 = 1. \tag{2.2}$$

[4] This is strictly true for the discrete version of the theory we will use in the following. However, one can guess, as discussed in [3], that it should be true also in the continuous approach [4], on the ground that the continuous reduction process can be approximated as close as one wants by a discrete process.

[5] These inequalities were termed of Bell-like type, despite the fact that they have nothing to do with non-locality issues.

In such a way the reduction probability is higher around points where quantum mechanics predicts that there is a higher probability of localizing the particle. As a consequence, one must impose the normalization condition

$$\int_{-\infty}^{+\infty} \mathrm{d}\bar{q}\, \widetilde{f}^2(q-\bar{q}) = 1. \tag{2.3}$$

Equation (2.3) entails that $\widetilde{f}(x)$ has the dimensions of the inverse of the square root of a distance. In order to use adimensional variables and an adimensional function, we introduce a constant α having the dimensions of the inverse square of a length, writing

$$\widetilde{f}(q-\bar{q}) = \alpha^{1/4} f[\sqrt{\alpha}(q-\bar{q})]. \tag{2.4}$$

Then equation (2.3) becomes

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, f^2(\sqrt{\alpha}q - x) = 1. \tag{2.5}$$

It is thus easy to appreciate the physical meaning of the parameter α; since, by assumption, $\widetilde{f}(y)$ goes rapidly to zero for large y, it follows that $f(\sqrt{\alpha}y)$ is significantly different from zero only when $y \simeq 1/\sqrt{\alpha}$. Thus, the latter parameter represents the characteristic localization length.

In passing to the formulation of the GRW theory for statistical operators ρ, we shall denote by $T[\rho]$ the action of the process (2.1). Let $\rho = |\psi\rangle\langle\psi|$; since the spatial distribution probability of the localization events is given by $\|\phi_{\bar{q}}\|^2$, the latter quantity represents the weight of each component of the post-localization statistical ensemble described by $T[\rho]$. Then we get

$$\langle q|T[\rho]|q'\rangle = \int_{-\infty}^{+\infty} \mathrm{d}\bar{q}\, \|\phi_{\bar{q}}\|^2 \psi_{\bar{q}}(q)\psi_{\bar{q}}^{\star}(q') \tag{2.6}$$

$$\langle q|T[\rho]|q'\rangle = F[\sqrt{\alpha}(q-q')]\langle q|\rho|q'\rangle, \tag{2.7}$$

where

$$F(\sqrt{\alpha}x) = \int_{-\infty}^{+\infty} \mathrm{d}y\, f(y) f(y+\sqrt{\alpha}x) \tag{2.8}$$

is a real, positive and adimensional function which is called the *jump function*.

Putting $y' = y + \sqrt{\alpha}x$ into equation (2.8), one immediately sees that F is an even function of its variable; also, using equation (2.5) one gets $F(0) = 1$.

Moreover, since $f[\sqrt{\alpha}(q-q')]$ is practically zero for $|q-q'| > 1/\sqrt{\alpha}$, equation (2.8) shows that $F[\sqrt{\alpha}(q-q')]$ is different from zero only for $|q-q'| < 1/\sqrt{\alpha}$.

Further, being $f(x)$ an even function of x, as shown in [3], the derivative of F turns out to be

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \int_{0}^{+\infty} \mathrm{d}y[f(y-x) - f(y+x)]\frac{\mathrm{d}f}{\mathrm{d}y}. \tag{2.9}$$

It follows that $\mathrm{d}F/\mathrm{d}x|_{x=0} = 0$. For $x > 0$, since $|y-x| \leqslant |y+x|$, being $y \in [0, +\infty)$, the assumed properties of f ensure $f(y-x) \geqslant f(y+x)$. Being moreover $\mathrm{d}f/\mathrm{d}y \leqslant 0$ for $y \geqslant 0$, $F(x)$ turns out to be a decreasing function of x for $x > 0$. Using the same arguments, one can show that $F(x)$ is an increasing function of x for $x < 0$.

Concluding, the hypotheses (2) on the function f (whichever it be) make the function $F[\sqrt{\alpha}(q-q')]$ an even function of $q-q'$, having a maximum for $q = q'$, and rapidly decreasing for $|q-q'| \geqslant 2/\sqrt{\alpha}$.

The analysis of [3] shows moreover that almost all functions $f(\sqrt{\alpha}q)$ (defining the jump function through equation (2.8)) having the mathematical features indicated above are acceptable, reproducing exactly the same characteristics of the model. Being the efficiency of the *jump* dependent only on the value of the parameter α, the choice of a Gaussian function for f becomes a pure matter of mathematical convenience, and we will use it in the following.

Finally, using equation (2.6), the process $\rho \mapsto T[\rho]$ can be written explicitly as follows: since

$$\langle q|T[\rho]|q'\rangle = \int_{-\infty}^{+\infty} \mathrm{d}x\, f(\sqrt{\alpha}q - x)\langle q|\rho|q'\rangle f(\sqrt{\alpha}q' - x), \tag{2.10}$$

one obtains

$$T[\rho] = \int_{-\infty}^{+\infty} \mathrm{d}x\, f(\sqrt{\alpha}\hat{q} - x)\rho f(\sqrt{\alpha}\hat{q} - x), \tag{2.11}$$

where $\hat{q}$ denotes the position operator ($\hat{q}\psi(x) = x\psi(x)$).

3. Ford and Lewis approach

In [5] Ford and Lewis considered a one-dimensional system whose positions $x_i \in \mathbf{R}$ at different times t_i form a stochastic process with joint probability distributions

$$W(x_1t_1, \ldots, x_nt_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$$

for its position to be found inside intervals $(x_i, x_i + \mathrm{d}x_i)$ at times t_i, with $t_1 < t_2 < \cdots < t_n$. That this is indeed the case is ascertained through fuzzy position measurements of finite accuracy. The effects of the latter measurements are quite similar to those forming the basis of the GRW model; localization within an interval I_i at time t_i amounts to the following process on the system density matrix $\rho(t_i)$ at time $t = t_i$,

$$\rho(t_i) \to \int_{I_i} \mathrm{d}x_i\, \alpha(\hat{x} - x_i)\rho(t_i)\alpha(\hat{x} - x_i), \tag{3.1}$$

where the operator α has the form

$$\alpha(\hat{x} - x_i) = \frac{1}{\sqrt[4]{\pi\sigma_i^2}}\,\mathrm{e}^{-\frac{(\hat{x}-x_i)^2}{2\sigma_i^2}}, \tag{3.2}$$

and σ_i represents the experimental width of the ith measurement, i.e. its accuracy. In between two subsequent fuzzy localization processes the system is assumed to evolve freely according to the Schrödinger time evolution $U(t)$ and the processes in equation (3.1) are considered as instantaneous with respect to $U(t)$.

By means of the operators

$$\alpha(m) := U^\dagger(t_m - t_0)\alpha(\hat{x} - x_m)U(t_m - t_0), \tag{3.3}$$

where $U(t_m - t_0)$ is the time evolution operator, Ford and Lewis showed that the joint probabilities can be written as

$$W(x_1t_1, \ldots, x_nt_n) = \mathrm{Tr}\{\alpha(n)\cdots\alpha(1)\rho(t_0)\alpha(1)\cdots\alpha(n)\}. \tag{3.4}$$

This expression is the quantum counterpart of a distribution of a classical ensemble of tracks, each having its own probability of occurrence. In such a case, it represents the fraction of tracks subjected to the process (3.1) at times $t_1 < t_2 < \cdots < t_n$.

In the classical case the joint probabilities obey three conditions:

(i) $W(x_1t_1, \ldots, x_nt_n) \geqslant 0$;

(ii) $W(x_1t_1, \ldots, x_it_i, \ldots, x_jt_j, \ldots, x_nt_n) = W(x_1t_1, \ldots, x_jt_j, \ldots, x_it_i, \ldots, x_nt_n)$;

(iii) $W(x_1t_1, \ldots, x_{n-1}t_{n-1}) = \int_{-\infty}^{+\infty} \mathrm{d}x_n\, W(x_1t_1, \ldots, x_nt_n)$.

Condition (iii) is the marginal form of a more general relation

$$W(x_1t_1, \ldots, x_{l-1}t_{l-1}, x_{l+1}t_{l+1}, \ldots, x_nt_n) = \int_{-\infty}^{+\infty} \mathrm{d}x_l\, W(x_1t_1, \ldots, x_nt_n), \tag{3.5}$$

which is valid if condition (ii) is fulfilled. In turn, the validity of condition (ii), which is a symmetry condition, descends from the fact that a classical measurement can be performed without disturbance for the free time evolution of the system, so that the time tracks are not affected by position measurements. This is obviously not true in the quantum case (in fact, the operators $\alpha(l)$ do not commute with each other), and therefore the symmetry condition is not valid. As a consequence, condition (3.5) is no longer satisfied. Still, it remains valid in its marginal form. This is due to the invariance of the trace under cyclic permutations and to the fact that

$$\frac{1}{\sqrt{\pi\sigma_n{}^2}} \int_{-\infty}^{+\infty} \mathrm{d}x_n\, \mathrm{e}^{-\frac{(q-x_n)^2}{\sigma_n^2}} = 1. \tag{3.6}$$

Starting from these considerations, one can introduce typical inequalities in the following way. Let the joint probabilities $\widetilde{W}(1, 2, \ldots, n)$ that the observable $\hat{x}$ takes values within finite intervals I_i at times t_i be denoted by

$$\widetilde{W}(1, 2, \ldots n) = \int_{I_1} \mathrm{d}x_1 \ldots \int_{I_n} \mathrm{d}x_n\, W(x_1t_1, \ldots, x_nt_n). \tag{3.7}$$

We denote also by $\widetilde{W}(1, 2, \ldots, \bar{m}, \ldots, n)$ the probability that $\hat{x}$ takes values outside I_m at time t_m. Then, in the classical case, in which equation (3.5) holds, one has

$$\widetilde{W}(1, \ldots, m-1, m+1, \ldots, n) = \widetilde{W}(1, \ldots, m, \ldots, n) + \widetilde{W}(1, \ldots, \bar{m}, \ldots, n). \tag{3.8}$$

Then, considering tracks with three measurements, one derives

$$\begin{aligned} \widetilde{W}(2, 3) &= \widetilde{W}(1, 2, 3) + \widetilde{W}(\bar{1}, 2, 3), \\ \widetilde{W}(1, 3) &= \widetilde{W}(1, 2, 3) + \widetilde{W}(1, \bar{2}, 3), \\ \widetilde{W}(1, \bar{2}) &= \widetilde{W}(1, \bar{2}, 3) + \widetilde{W}(1, \bar{2}, \bar{3}). \end{aligned} \tag{3.9}$$

Since the quantities $\widetilde{W}$ are positive, it follows that

$$\widetilde{W}(1, 3) \leqslant \widetilde{W}(2, 3) + \widetilde{W}(1, \bar{2}). \tag{3.10}$$

This relation is valid in the classical case, but not in the quantum one. Indeed, in the quantum case, the last of relations (3.9) remains valid, but the first two are not satisfied, due to the failure of relation (3.5) as a consequence of non-commutativity. Some extra conditions are needed in order to satisfy classical laws, as, for example, the decoherence condition in the decoherent histories approach [7].

This fact connects the above inequalities to Bell-like ones which are violated by quantum mechanical systems. Only, there are no non-local correlations between spatially separated systems here, rather quantum correlations between the positions of a same system at different times.

4. The Schrödinger case

Instead of taking into consideration inequality (3.10), we deal with a simpler but enlightening case, which permits to clarify the different roles of the Schrödinger and of the GRW time evolutions (see (5.1) below), avoiding too cumbersome calculations.

We study the violation of equality (3.8) in the particular case $n = 2$, i.e., in the case of the Schrödinger evolution of a system represented at time t_0 by a statistical operator ρ_0 and subjected to only two measurements at times t_1 and t_2, with $t_1 < t_2$. In particular, we will focus upon the difference

$$D = \widetilde{W}(2) - [\widetilde{W}(1,2) + \widetilde{W}(\bar{1},2)]. \tag{4.1}$$

We denote by $\Sigma^S_{t-t'}$ the Schrödinger time evolution, with Hamiltonian $H = \hat{p}^2\frac{1}{2m}$, sending $\rho^S(t')$ into $\rho^S(t)$,

$$\rho^S(t) = \Sigma^S_{t-t'}[\rho^S(t')] = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t-t')}\rho^S(t')\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}H(t-t')}, \tag{4.2}$$

and by $T^\sigma_z\rho$ the effect of the measurement around z with accuracy σ,

$$T^\sigma_z[\rho] = \alpha(\hat{q} - z)\rho\alpha(\hat{q} - z), \tag{4.3}$$

where $\alpha(\hat{q} - z)$ is given by equation (3.2). Then, equation (3.7) with $n = 2$ explicitly reads

$$\widetilde{W}(2) = \int_{I_2} \mathrm{d}x_2\, \mathrm{Tr}\left\{T^{\sigma_2}_{x_2} \circ \Sigma^S_{t_2-t_1}[\rho^S(t_1)]\right\} \tag{4.4}$$

and

$$\widetilde{W}(1,2) + \widetilde{W}(\bar{1},2) = \int_{I_2} \mathrm{d}x_2 \int_{-\infty}^{+\infty} \mathrm{d}x_1\, \mathrm{Tr}\left\{T^{\sigma_2}_{x_2} \circ \Sigma^S_{t_2-t_1} \circ T^{\sigma_1}_{x_1}[\rho^S(t_1)]\right\}. \tag{4.5}$$

Using equation (4.4) a simple calculation gives

$$\begin{aligned}\widetilde{W}(2) &= \frac{1}{\sqrt{\pi\sigma_2{}^2}} \int_{I_2} \mathrm{d}x_2 \int_{-\infty}^{+\infty} \mathrm{d}q\, \mathrm{e}^{-\frac{(q-x_2)^2}{\sigma_2^2}} \langle q|\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t_2-t_1)}\rho^S(t_1)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}H(t_2-t_1)}|q\rangle \\ &= \frac{1}{\sqrt{\pi\sigma_2{}^2}} \int_{I_2} \mathrm{d}x_2 \int_{-\infty}^{+\infty} \mathrm{d}q\, \mathrm{e}^{-\frac{(q-x_2)^2}{\sigma_2^2}} \langle q|\rho^S(t_2)|q\rangle. \end{aligned} \tag{4.6}$$

In order to calculate explicitly the quantity $\widetilde{W}(1,2) + \widetilde{W}(\bar{1},2)$ through equation (4.5), the following relation makes the procedure straightforward. Using equations (4.3) and (3.2) we obtain

$$\begin{aligned}\langle q_1|T^\sigma_x[\rho]|q_2\rangle &= \frac{1}{\sqrt{\pi\sigma_2{}^2}}\,\mathrm{e}^{-\frac{1}{2\sigma^2}[(q_1-x)^2+(q_2-x)^2]}\langle q_1|\rho|q_2\rangle \\ &= \frac{1}{\sqrt{\pi\sigma_2{}^2}}\,\mathrm{e}^{-\frac{(q_1-q_2)^2}{4\sigma^2}}\,\mathrm{e}^{-\frac{1}{\sigma^2}(x-\frac{q_1+q_2}{2})^2}\langle q_1|\rho|q_2\rangle. \end{aligned} \tag{4.7}$$

It follows that

$$\langle q_1|\left(\int_{-\infty}^{+\infty}\mathrm{d}x\, T^\sigma_x[\rho]\right)|q_2\rangle = \mathrm{e}^{-\frac{(q_1-q_2)^2}{4\sigma^2}}\langle q_1|\rho|q_2\rangle. \tag{4.8}$$

Introducing a delta function $\delta(q - q_1 + q_2)$, one obtains

$$\langle q_1|\left(\int_{-\infty}^{+\infty}\mathrm{d}x\, T^\sigma_x[\rho]\right)|q_2\rangle = \int_{-\infty}^{+\infty}\mathrm{d}q\,\delta(q-q_1+q_2)\,\mathrm{e}^{-\frac{q^2}{4\sigma^2}}\langle q_1|\rho|q_2\rangle \tag{4.9}$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}p\int_{-\infty}^{+\infty}\mathrm{d}q\,\mathrm{e}^{-\frac{q^2}{4\sigma^2}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(q-q_1+q_2)p}\langle q_1|\rho|q_2\rangle, \tag{4.10}$$

whence integration with respect to q yields

$$\langle q_1|\left(\int_{-\infty}^{+\infty}\mathrm{d}x\,T_x^{\sigma}[\rho]\right)|q_2\rangle = \sqrt{\frac{\sigma^2}{\pi\hbar^2}}\int_{-\infty}^{+\infty}\mathrm{d}p\,\mathrm{e}^{-\frac{\sigma^2p^2}{\hbar^2}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p(q_2-q_1)}\langle q_1|\rho|q_2\rangle$$
$$= \sqrt{\frac{\sigma^2}{\pi\hbar^2}}\int_{-\infty}^{+\infty}\mathrm{d}p\,\mathrm{e}^{-\frac{\sigma^2p^2}{\hbar^2}}\,\langle q_1|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\,\rho\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{q}}|q_2\rangle. \tag{4.11}$$

Therefore, one finally gets

$$\int_{-\infty}^{+\infty}\mathrm{d}x\,T_x^{\sigma}[\rho] = \sqrt{\frac{\sigma^2}{\pi\hbar^2}}\int_{-\infty}^{+\infty}\mathrm{d}p\,\mathrm{e}^{-\frac{\sigma^2p^2}{\hbar^2}}\,\mathrm{e}^{-\frac{i}{\hbar}p\hat{q}}\,\rho\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{q}}. \tag{4.12}$$

We note that equation (4.12) is valid independently of the type of evolution which the ρ-matrix is subjected to in between two subsequent fuzzy localizations. Inserting equation (4.12) into equation (4.5), one obtains

$$\widetilde{W}(1,2)+\widetilde{W}(\bar{1},2) = \sqrt{\frac{{\sigma_1}^2}{\pi\hbar^2}}\int_{I_2}\mathrm{d}x_2\int_{-\infty}^{+\infty}\mathrm{d}p\,\mathrm{e}^{-\frac{{\sigma_1}^2p^2}{\hbar^2}}\,\mathrm{Tr}\big\{T_{x_2}^{\sigma_2}\circ\Sigma_{t_2-t_1}^{S}\big[\big(\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\rho^S(t_1)\,\mathrm{e}^{\frac{i}{\hbar}p\hat{q}}\big)\big]\big\}. \tag{4.13}$$

Further, since it was assumed the time evolution $U(t)$ to be generated by the free Hamiltonian $H=\frac{1}{2m}\hat{p}^2$, the commutation relation

$$\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{(t_2-t_1)}{2m}\hat{p}}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{(t_2-t_1)}{2m}(\hat{p}-p)^2} \tag{4.14}$$

yields

$$\Sigma_{t_2-t_1}^{S}\big[\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\rho^S(t_1)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{q}}\big] = \mathrm{e}^{-\frac{i}{\hbar}p\hat{q}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{p(t_2-t_1)}{2m}\hat{p}}\,\Sigma_{t_2-t_1}^{S}[\rho^S(t_1)]\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{p(t_2-t_1)}{2m}\hat{p}}\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}. \tag{4.15}$$

Thus, using equation (4.14), the fact that

$$\mathrm{e}^{-\frac{(\hat{q}-x_2)^2}{2\sigma_2^2}}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{p(t_2-t_1)}{m}\hat{p}} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\frac{p(t_2-t_1)}{m}\hat{p}}\,\mathrm{e}^{-\frac{1}{2\sigma_2^2}(\hat{q}-x_2-\frac{p(t_2-t_1)}{m})^2} \tag{4.16}$$

and the group composition law $\Sigma_{t_2-t_1}^{S}[\rho^S(t_1)]=\rho^S(t_2)$, one gets

$$\mathrm{Tr}\big\{T_{x_2}^{\sigma_2}\circ\Sigma_{t_2-t_1}^{S}\big[\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}p\hat{q}}\rho^S(t_1)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}p\hat{q}}\big]\big\} = \mathrm{Tr}\big\{T_{x_2+\frac{p}{m}(t_2-t_1)}^{\sigma_2}[\rho^S(t_2)]\big\}. \tag{4.17}$$

Therefore equation (4.13) becomes

$$\widetilde{W}(1,2)+\widetilde{W}(\bar{1},2) = \sqrt{\frac{{\sigma_1}^2}{\pi\hbar^2}}\int_{I_2}\mathrm{d}x_2\int_{-\infty}^{+\infty}\mathrm{d}p\,\mathrm{e}^{-\frac{{\sigma_1}^2p^2}{\hbar^2}}\,\mathrm{Tr}\big\{T_{x_2+\frac{p}{m}(t_2-t_1)}^{\sigma_2}[\rho^S(t_2)]\big\}. \tag{4.18}$$

Finally, integration over the variable p yields

$$\widetilde{W}(1,2)+\widetilde{W}(\bar{1},2) = \frac{1}{\sqrt{\pi\,\Sigma_2^2}}\int_{I_2}\mathrm{d}x_2\int_{-\infty}^{+\infty}\mathrm{d}q\,\mathrm{e}^{-\frac{1}{\Sigma_2^2}(q-x_2)^2}\,\langle q|\rho^S(t_2)|q\rangle, \tag{4.19}$$

where

$$\Sigma_2 = \sqrt{\sigma_2^2+\frac{\hbar^2(t_2-t_1)^2}{m^2\sigma_1^2}}. \tag{4.20}$$

By rewriting equation (4.4) as

$$\widetilde{W}(2) = \int_{I_2}\mathrm{d}x_2\,\mathrm{Tr}\big\{T_{x_2}^{\sigma_2}[\rho^S(t_2)]\big\}, \tag{4.21}$$

comparison of equations (4.6) and (4.19) allows one to rewrite

$$\widetilde{W}(1,2) + \widetilde{W}(\bar{1},2) = \int_{I_2} \mathrm{d}x_2\, \mathrm{Tr}\big\{T_{x_2}^{\Sigma_2}[\rho^S(t_2)]\big\}. \tag{4.22}$$

Thus, one sees that equation (4.22) differs from equation (4.21) in that the fuzzy localization at time t_2 is performed with accuracy Σ_2 instead of σ_2: this is the only effect brought about by the action at $t = t_1$ of the non-selective fuzzy localization process with accuracy σ_1.

We now proceed to a detailed investigation of the difference D in equation (4.1) between the probabilities of the system being localized within I_2 when the non-selective process at t_1 occurs or not. This difference vanishes in two cases:

- if $I_2 \gg \Sigma_2$, in fact in this case the integral over x_2 can be extended from $-\infty$ to $+\infty$, and equations (4.6) and (4.19) reduce both to $\mathrm{Tr}\{\rho^S(t_2)\}$ (i.e. to 1). This happens since in the case $I_2 \gg \Sigma_2$ the measurement at time t_2 is not effective;
- if $\frac{\hbar^2(t_2-t_1)^2}{m^2\sigma_1^2} \ll \sigma_2^2$, the physical meaning of this condition will be clarified in the following.

5. Joint probabilities and GRW evolution

Owing to the peculiarity of the QMSL evolution, using it instead of the Schrödinger one, the quantity D should turn out to be different from zero only in the microscopic case.

To see if this happens, we examine the behaviour of the difference D in the case of the GRW evolution, given by the equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{i}{\hbar}[H,\rho] - \lambda\{\rho - T(\rho)\}. \tag{5.1}$$

Equation (5.1) implies the existence of a map $\Sigma^{\lambda}_{t-t'}$, defined for $t' < t$.[6] Contrary to $\Sigma^S_{t-t'}$, it satisfies the relation $\Sigma^{\lambda}_{t-t''} = \Sigma^{\lambda}_{t-t'}\Sigma^{\lambda}_{t'-t''}$ only for ordered times $t'' < t' < t$. This is due to the fact that the evolution is irreversible, and, as such, equation (5.1) pertains to the class of the so-called quantum dynamical semigroup equations. We take then into consideration equations (4.5) and (4.6), substituting $\Sigma^S_{t-t'}$ with $\Sigma^{\lambda}_{t-t'}$ and calling $\rho^{\lambda}(t)$ the statistical operator evolved by Σ^{λ}_t. Owing to the discussion of section 1, we are free to choose the jump function which is more convenient from the mathematical point of view. It has been shown [9–11] that, if we use the Gaussian function introduced in [1], in all cases in which the Hamiltonian is at most quadratic in the position and momentum coordinates, the solution of equation (5.1) can be written as

$$\rho^{\lambda}(t) = \Sigma^{\lambda}_{t-t_0}[\rho^{\lambda}(t_0)] = \frac{1}{2\pi\hbar^2}\int_{\mathrm{R}^4} \mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}p\,\mathrm{d}\pi\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\pi x - p\xi)} F^{\alpha}_{\lambda}(\xi,\pi,t-t_0)$$
$$\times\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x\hat{p}-p\hat{q})}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}H(t-t_0)}\rho(t_0)\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}H(t-t_0)}\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}(x\hat{p}-p\hat{q})}, \tag{5.2}$$

where the modulating function in the integral is positive and reads

$$F^{\alpha}_{\lambda}(\xi,\pi,\tau) = \exp\left(-\lambda\left\{\tau - \int_0^{\tau}\mathrm{d}s\,\exp\left(-\frac{\alpha}{4}\left[\frac{\pi}{m}(\tau-s)-\xi\right]^2\right)\right\}\right). \tag{5.3}$$

In this case equations (4.6) and (4.21) become

$$\widetilde{W}(2) = \int_{I_2}\mathrm{d}x_2\,\mathrm{Tr}\big\{T^{\sigma_2}_{x_2}[\rho^{\lambda}(t_2)]\big\} \tag{5.4}$$

[6] The linearity of the map and the stochastic character of the evolution ensure that the model is not affected with faster than light signalling (see [8]).

$$\widetilde{W}(1,2)+\widetilde{W}(\bar{1},2)=\int_{I_2}\mathrm{d}x_2\int_{-\infty}^{+\infty}\mathrm{d}x_1\,\mathrm{Tr}\big\{T_{x_2}^{\sigma_2}\circ\Sigma_{t_2-t_1}^{\lambda}\circ T_{x_1}^{\sigma_1}[\rho^{\lambda}(t_1)]\big\}. \tag{5.5}$$

Since, as already remarked, equation (4.12) holds independently of the time evolution in between subsequent fuzzy measurements, equation (5.5) becomes

$$\widetilde{W}(1,2)+\widetilde{W}(\bar{1},2)=\sqrt{\frac{{\sigma_1}^2}{\pi\hbar^2}}\int_{I_2}\mathrm{d}x_2\int_{-\infty}^{+\infty}\mathrm{d}p\,\mathrm{e}^{-\frac{{\sigma_1}^2p^2}{\hbar^2}}\,\mathrm{Tr}\big\{T_{x_2}^{\sigma_2}\circ\Sigma_{t_2-t_1}^{\lambda}\big[\mathrm{e}^{-\frac{i}{\hbar}p\hat{q}}\rho^{\lambda}(t_1)\,\mathrm{e}^{\frac{i}{\hbar}p\hat{q}}\big]\big\}. \tag{5.6}$$

A calculation similar to the one in the Schrödinger case yields

$$\mathrm{Tr}\big\{T_{x_2}^{\sigma_2}\circ\Sigma_{t_2-t_1}^{\lambda}\big[\mathrm{e}^{-\frac{i}{\hbar}p\hat{q}}\rho^{\lambda}(t_1)\,\mathrm{e}^{\frac{i}{\hbar}p\hat{q}}\big]\big\}=\mathrm{Tr}\big\{T_{x_2+\frac{p}{m}(t_2-t_1)}^{\sigma_2}\circ\Sigma_{t_2-t_1}^{\lambda}[\rho^{\lambda}(t_1)]\big\}. \tag{5.7}$$

This is exactly the same relation which is valid in the Schrödinger case whence, since the type of evolution has no effect on the calculation, the result is

$$\widetilde{W}(1,2)+\widetilde{W}(\bar{1},2)=\int_{I_2}\mathrm{d}x_2\,\mathrm{Tr}\big\{T_{x_2}^{\Sigma_2}[\rho^{\lambda}(t_2)]\big\}. \tag{5.8}$$

Observe that equations (5.4) and (5.8) differ from equations (4.21) and (4.22) in that $\rho^{\lambda}(t_2)$ has taken the place of $\rho^{S}(t_2)$. Therefore, the conditions for the difference D to vanish are equal to the first two ones obtained in the Schrödinger case, i.e. $I_2 \gg \Sigma_2$ and $\frac{\hbar(t_2-t_1)}{m\sigma_1} \ll \sigma_2$. As a consequence, the effect of the GRW time evolution comes to the fore only if we are able to make explicit the dependence from the different evolutions of the ρ-matrices. Before doing this, a comment is necessary on the sufficient condition

$$\frac{\hbar(t_2-t_1)}{m\sigma_1} \ll \sigma_2. \tag{5.9}$$

Physical considerations suggest that D can vanish only if the spread in position produced by the fuzzy non-selective measurement at time t_1 cannot be revealed by the subsequent fuzzy position measurement at time t_2. One thus argues that this may happen only if that spread is smaller than σ_2 at time t_2. An indication whether this is the case or not comes from calculating the mean square-deviation,

$$(\Delta q)_{t_2}^2=\langle\hat{q}^2\rangle_{t_2}-\langle\hat{q}\rangle_{t_2}^2, \tag{5.10}$$

where

$$\langle\hat{q}^n\rangle_{t_2}=\int_{-\infty}^{+\infty}\mathrm{d}x_1\,\mathrm{Tr}\big\{\hat{q}^n\,\Sigma_{t_2-t_1}\circ T_{x_1}^{\sigma_1}[\rho(t_1)]\big\}, \tag{5.11}$$

$\Sigma_{t_2-t_1}$ being either $\Sigma_{t_2-t_1}^{\lambda}$ or $\Sigma_{t_2-t_1}^{S}$. The explicit calculations in [6] give

$$(\Delta q)_{t_2}^2=\mathrm{Tr}\{(\hat{q}-\langle\hat{q}\rangle)^2\rho(t_2)\}+\frac{1}{2}\left[\frac{\hbar(t_2-t_1)}{m\sigma_1}\right]^2, \tag{5.12}$$

where $\rho(t_2)$ is the density matrix evolved up to time t_2 either by the Schrödinger or by the GRW time evolution. In equation (5.12) the spreading effect of the dynamics appears clearly separated from the one due to the fuzzy measurement at time t_1, which amounts to the second term at the rhs,

$$\Delta q=\frac{1}{\sqrt{2}}\frac{\hbar(t_2-t_1)}{m\sigma_1}, \tag{5.13}$$

and is independent from the dynamics. Then, D can be zero only if Δq is smaller than σ_2, which leads exactly to condition (5.9).

6. The Gaussian case

In order to study in detail how the two types of evolution play a different role in shaping the time behaviour of the joint probabilities, we assume that the system under consideration is prepared at $t = 0$ in a Gaussian wavefunction centred around a point q_0:

$$\langle q|\psi\rangle = \frac{1}{\sqrt[2]{\pi\,\Delta^2}}\,\mathrm{e}^{-\frac{1}{\Delta^2}(q-q_0)^2}. \tag{6.1}$$

We first consider the system evolving in time according to the Schrödinger evolution; then, the diagonal matrix element of ρ in position representation in equations (4.6) and (4.19) turns out to be

$$\langle q|\rho^S(t_2)|q\rangle = \langle q|\psi(t_2)\rangle\langle\psi(t_2)|q\rangle = \left|\langle q|\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\frac{t_2}{2m}\hat{p}^2}|\psi\rangle\right|^2. \tag{6.2}$$

The use of a Gaussian wavefunction is particularly convenient due to the fact that it remains Gaussian under the Schrödinger time evolution:

$$\langle q|\rho^S(t_2)|q\rangle = \frac{1}{\sqrt{\pi\,\Delta_2^2}}\,\mathrm{e}^{-\frac{1}{\Delta_2^2}(q-q_0)^2}, \qquad \Delta_2^2 = \Delta^2 + \frac{\hbar^2 t_2^2}{m^2\Delta^2}. \tag{6.3}$$

Inserting equation (6.3) into equations (4.6) and (4.19), integration over q yields

$$\widetilde{W}(2) = \frac{1}{\sqrt{\pi\left(\sigma_2^2+\Delta_2^2\right)}}\int_{I_2}\mathrm{d}x_2\,\mathrm{e}^{-\frac{1}{\sigma_2^2+\Delta_2^2}(x_2-q_0)^2} \tag{6.4}$$

$$\widetilde{W}(1,2) + \widetilde{W}(\bar{1},2) = \frac{1}{\sqrt{\pi\left(\Sigma_2^2+\Delta_2^2\right)}}\int_{I_2}\mathrm{d}x_2\,\mathrm{e}^{-\frac{1}{\Sigma_2^2+\Delta_2^2}(x_2-q_0)^2}, \tag{6.5}$$

where Σ_2 is given by equation (4.20). Comparing equation (6.4) with equation (6.5) shows that the sufficient conditions for $D = 0$ are now

$$I_2 \gg \sqrt{\Sigma_2^2+\Delta_2^2} = \sqrt{\sigma_2^2 + \frac{\hbar^2(t_2-t_1)^2}{m^2\sigma_1^2} + \Delta^2 + \frac{\hbar^2 t_2^2}{m^2\Delta^2}} \tag{6.6}$$

$$\frac{\hbar^2(t_2-t_1)^2}{m^2\sigma_1^2} \ll \sigma_2^2 \tag{6.7}$$

$$\Delta^2 + \frac{\hbar^2 t_2^2}{m^2\Delta^2} \gg \frac{\hbar^2(t_2-t_1)^2}{m^2\sigma_1^2}. \tag{6.8}$$

The first condition is satisfied when the fuzzy localization interval far exceeds the spreading in position at time t_2 so that, in the integration, one can assume $I_2 \simeq \mathbf{R}$. If the first condition is not fulfilled, the second one corresponds to the impossibility that the spread due to the fuzzy measurement at time t_1 be felt by the fuzzy measurement at time t_2. Finally, if also the second condition is not fulfilled, D can nevertheless vanish because the spreading due to the fuzzy measurement at time t_1 is blurred by the spreading due to the Schrödinger time evolution as embodied by the last condition above.

Let us now consider the GRW evolution. Equation (5.2) gives

$$\begin{aligned}\langle q|\rho^\lambda(t_2)|q\rangle &= \frac{1}{2\pi\hbar^2}\int_{\mathbf{R}^4}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}p\,\mathrm{d}\pi\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(\pi x - p\xi)}F_\lambda^\alpha(\xi,\pi,t_2)\left|\langle q|\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x\hat{p}-p\hat{q})}|\psi(t_2)\rangle\right|^2 \\ &= \frac{1}{2\pi\hbar^2}\int_{\mathbf{R}^4}\mathrm{d}x\,\mathrm{d}\xi\,\mathrm{d}p\,\mathrm{d}\pi\,F^{\alpha_\lambda}(\xi,\pi,t_2)|\langle q+x|\psi(t_2)\rangle|^2.\end{aligned} \tag{6.9}$$

Using the $\delta(\xi)$ arising from the integration over p, we obtain

$$\langle q|\rho^{\lambda}(t_2)|q\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}\pi\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\pi x}F^{\alpha_\lambda}(0,\pi,t_2)\frac{1}{\sqrt{\pi\Delta_2^2}}\,\mathrm{e}^{-\frac{1}{\Delta_2^2}(q+x-q_0)^2}. \tag{6.10}$$

Inserting this expression into equations (4.6) and (4.19) (that are valid also for the GRW evolution, in which case $\rho^S(t_2)$ is replaced by $\rho^{\lambda}(t_2)$), one finally gets

$$\widetilde{W}(2) = \frac{1}{2\pi\hbar}\int_{I_2}\mathrm{d}x_2\int_{-\infty}^{+\infty}\mathrm{d}p F_{\lambda}^{\alpha}(0,\pi,t_2)\,\mathrm{e}^{-\frac{\Delta_2^2+\sigma_2^2}{4\hbar^2}p^2}\,\mathrm{e}^{\frac{i}{\hbar}(q_0-x_2)p}, \tag{6.11}$$

$$\widetilde{W}(1,2)+\widetilde{W}(\bar{1},2) = \frac{1}{2\pi\hbar}\int_{I_2}\mathrm{d}x_2\int_{-\infty}^{+\infty}\mathrm{d}p F_{\lambda}^{\alpha}(0,\pi,t_2)\,\mathrm{e}^{-\frac{\Delta_2^2+\Sigma_2^2}{4\hbar^2}p^2}\,\mathrm{e}^{\frac{i}{\hbar}(q_0-x_2)p}. \tag{6.12}$$

We remark that for $\alpha = 0$ (i.e., no spontaneous localization process) equations (6.11) and (6.12) become equal to equations (6.4) and (6.5), respectively, as must be. The sufficient conditions for $D = 0$ become $I_2 \gg \Sigma_2$ and inequality (6.8).

Suppose now $D \neq 0$ in the case of the Schrödinger evolution; this means that conditions (6.6)–(6.8) are not satisfied, i.e., I_2 is not greater than $\sqrt{\Sigma_2^2+\Delta_2^2}$ and

$$\frac{\hbar^2(t_2-t_1)^2}{m^2\sigma_1^2} \simeq \sigma_2^2, \tag{6.13}$$

$$\Delta_2^2 \simeq \frac{\hbar^2(t_2-t_1)^2}{m^2\sigma_1^2} \simeq \sigma_2^2. \tag{6.14}$$

In such a case, for the GRW evolution, equations (6.11) and (6.12) give

$$|D| = \frac{1}{2\pi\hbar}\left|\int_{I_2}\mathrm{d}x_2\int_{-\infty}^{+\infty}\mathrm{d}p F_{\lambda}^{\alpha}(0,p,t_2)\,\mathrm{e}^{-\frac{\sigma_2^2}{2\hbar^2}p^2}\left(1-\mathrm{e}^{-\frac{\sigma_2^2}{4\hbar^2}p^2}\right)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(q_0-x_2)p}\right|. \tag{6.15}$$

Direct inspection of equation (6.15) indicates that $|D|$ differs from zero only if $1-\mathrm{e}^{-\frac{\sigma_2^2}{4\hbar^2}p^2}$ and $\mathrm{e}^{-\frac{\sigma_2^2}{2\hbar^2}p^2}$ are significantly different from zero. This happens if $p \geqslant 2\hbar/\sigma_2$ and $p \leqslant \sqrt{2}\hbar/\sigma_2$, i.e. for $p \simeq \hbar/\sigma_2$.

Without too much restriction, one can assume $\sigma_1 \simeq \sigma_2 = \sigma$, whence equation (6.13) gives

$$\sigma \simeq \sqrt{\frac{\hbar(t_2-t_1)}{m}}. \tag{6.16}$$

For a microscopic system with $m = 10^{-24}g$ and for a reasonable time interval between subsequent fuzzy measurements of the order of 10^{-3}s one obtains the physically reasonable value

$$\sigma \simeq 10^{-3}\,\mathrm{cm}. \tag{6.17}$$

However, if the system is macroscopic, with $N \simeq 10^{24}$ so that its mass $m \simeq 1g$, then

$$\sigma_1 = \sigma_2 \simeq 10^{-15}\,\mathrm{cm}, \tag{6.18}$$

a constraint that would ask for a localization accuracy smaller than the atomic dimensions, clearly an unfeasible condition.

Therefore, in the microscopic case D may be seen not vanishing and thus classical behaviour excluded both under the Schrödinger and GRW time evolutions. On the other hand, for macroscopic particles, D may be seen not to vanish in the case of the Schrödinger time evolution by choosing physically reasonable fuzzy localization accuracies, while these latter get out of reach if one pretends to expose $D \neq 0$ when the time evolution is of GRW-type.

7. Conclusions

The preceding analysis is a nice example of how the GRW mechanism works in the case of the evolution of a one-dimensional quantum system subjected to random position measurements. For such a system the related joint probabilities are linked by conditions valid classically, but violated by quantum mechanics. This fact allows one to derive a sort of Bell inequalities, fulfilled in the classical case, but violated in the quantum one. If the GRW model replaces the standard Schrödinger time evolution, it happens that these Bell-type inequalities can always be violated if the system is microscopic, exactly as in standard quantum mechanics; on the contrary, they cannot be violated in any reasonable sense by a macroscopic particle whose behaviour can in this respect be considered classical.

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